Int J Theor Phys (2008) 47: 2566–2575
DOI 10.1007/s10773-008-9690-5

Gravitational Aharonov-Bohm Effect

Michael Heller · Zdzisław Odrzygóźdź · Leszek Pysiak · Wiesław Sasin

Received: 8 December 2007 / Accepted: 13 February 2008 / Published online: 22 February 2008

Abstract We study a purely gravitational Aharonov-Bohm effect. The space-time curvature is concentrated in the quasiregular singularity of a cosmic string, outside of which space-time is (locally) flat. The symmetries of this field configuration are described by the groupoid symmetries rather than by the usual group symmetries. The groupoid in question is formed by homotopy classes of piecewise smooth paths in the cosmic string region. A gravitational counterpart of the Aharonov-Bohm effect occurs if the symmetry of the system, with respect to the groupoid action, is broken down.

Keywords Gravitational Aharonov-Bohm effect · Quasiregular singularity · Fundamental groupoid

1 Introduction

There are several versions of what is called Aharonov-Bohm effect. The most commonly referred to is the one in which a charged particle travels around a solenoid that encloses a magnetic field, but produces no field outside in the region in which the particle travels. In spite of this, the particle exhibits a phase shift that can be experimentally observed. An analogous effect is expected to occur in the case of gravitational field [1, 2, 4]. The required field configuration is when, for example, space-time curvature is concentrated in a cosmic string and is zero outside of it. More generally, by gravitational Aharonov-Bohm effect authors sometimes understand situations in which motions, in locally flat regions of space-time, exhibit behavior different from that in the Minkowski space-time [3].

M. Heller (✉)
Vatican Observatory, 00120 Vatican, Vatican City State
e-mail: mheller@wsd.tarnow.pl

Present address: M. Heller
ul. Powstańców Warszawy 13/94, 33-100 Tarnów, Poland

Z. Odrzygóźdź · L. Pysiak · W. Sasin
Department of Mathematics and Information Science, Warsaw University of Technology,
Plac Politechniki 1, 00-661 Warsaw, Poland

In the literature, various such effects were considered, for instance: in a weak field approximation [8], with the use of either Klein-Gordon equation [3] or Dirac's equation [12], or as a combination of gravitational and magnetic effects [14] (this is by no means a fair sample of works done in this field). In the present paper, we focus on the purely gravitational analogue of the Aharonov-Bohm effect caused by the space-time curvature concentrated in the quasiregular singularity of a cosmic string. Our main investment into the problem is that the symmetries of the investigated configuration are not the usual group symmetries, but rather generalized groupoid symmetries. We further assume only very mild properties of the detecting particle: its quantum states are described by elements of a Hilbert space that naturally appears in our formalism, and its position observable is given by the usual multiplicative operator on this Hilbert space.

In Sect. 2, we recall the groupoid definition and some related concepts just to fix notation and to make the paper self-contained. In Sect. 3, we present the method of representing the groupoid structure on a Hilbert space, and its "reduction" to functions defined on the base space. After reminding, in Sect. 4, the structure of the fundamental groupoid of piecewise smooth paths on a manifold, we construct, in Sect. 5, such a groupoid around the quasiregular singularity of a cosmic string and, by using the method presented in Sect. 2, we formulate the condition which, if satisfied, the phase shift occurs rendering the position measurement impossible. Purely gravitational Aharonov-Bohm effect occurs only if the field configuration around the cosmic string is not symmetric with respect to the action of the considered groupoid.

2 Preliminaries

Let us start with the groupoid definition [10]:

Definition 1 A groupoid Γ over Γ_0, or a groupoid with base Γ_0, is a 7-tuple $(\Gamma, d, r, \varepsilon, i, m; \Gamma_0)$ consisting of the following elements:

- the sets Γ and Γ_0,
- the surjections $(d, r) : \Gamma \to \Gamma_0$ called the source and the target maps, respectively,
- the injection $\varepsilon : \Gamma_0 \to \Gamma$, $\gamma \mapsto \epsilon(\gamma)$, called the identity section,
- the map $i : \Gamma \to \Gamma$, $\gamma \mapsto i(\gamma) = \gamma^{-1}$, called the inversion map.

A composition law is defined $m : \Gamma^{(2)} \to \Gamma$, $(\gamma, \xi) \mapsto m(\gamma, \xi) = \gamma\xi$, with the domain $\Gamma^{(2)} := \{(\gamma, \xi) \in \Gamma \times \Gamma | d(\gamma) = r(\xi)\}$, such that the following axioms are satisfied:

(a) (associativity law) for arbitrary $\gamma, \xi, \eta \in \Gamma$ the triple product $(\gamma\xi)\eta$ is defined iff $\gamma(\xi\eta)$ is defined. In such a case we have

$$(\gamma\xi)\eta = \gamma(\xi\eta),$$

(b) (identities) for each $\gamma \in \Gamma$ we have

$$\varepsilon(r(\gamma))\gamma = \gamma\varepsilon(d(\gamma)),$$

(c) (inverses) for each $\gamma \in \Gamma$

$$\gamma i(\gamma) = \varepsilon(r(\gamma)),$$

$$i(\gamma)\gamma = \varepsilon(d(\gamma)).$$

A groupoid Γ over Γ_0 is also denoted by $(\Gamma, d, r; \Gamma_0)$ or by $\Gamma \overset{d}{\underset{r}{\rightrightarrows}} \Gamma_0$. In the categorial language, the set Γ_0 is called the set of objects, and the set Γ the set of morphisms. The mapping $\varepsilon : \Gamma_0 \to \Gamma$ associates the identity morphism id_x with every $x \in \Gamma_0$.

For each $\gamma \in \Gamma$ we define the sets

$$\Gamma^x = \{\gamma \in \Gamma | r(\gamma) = x, x \in \Gamma_0\},$$

$$\Gamma_x = \{\gamma \in \Gamma | d(\gamma) = x, x \in \Gamma_0\}.$$

Definition 2 A groupoid Γ over Γ_0 is said to be transitive if the map $(r, d) : \Gamma \to \Gamma_0 \times \Gamma_0$, given by

$$(r, d)(\gamma) = (r(\gamma), d(\gamma)), \quad \forall \gamma \in \Gamma,$$

is surjective.

The following proposition holds [5]:

Proposition 1 *If Γ is a groupoid over Γ_0, then*

(i) *for each $x \in \Gamma_0$, the set $\Gamma_x^x = \Gamma^x \cap \Gamma_x$ is a group, called the isotropy group at x,*
(ii) *if $d(\gamma) = x$, $r(\gamma) = y$ then the map $\omega : \Gamma_x^x \to \Gamma_y^y$, $\omega(\eta) = \gamma^{-1}\eta\gamma$, is an isomorphism of groups,*
(iii) *$\varepsilon(x)$ is the identity of Γ_x^x,*
(iv) *if Γ is transitive, then the isotropy groups of Γ are isomorphic.*

Let us notice that with each equivalence relation R on a set X, $R \subset X \times X$, we can associate the groupoid $R \overset{d}{\underset{r}{\rightrightarrows}} X$ and, *vice versa*, any groupoid Γ over a set X, where $(X = \Gamma_0)$, determines an equivalence relation $R \subset X \times X$. Indeed, let R be an equivalence relation on X. The groupoid associated with R is defined by regarding two projections $\pi_2(y, x) = x$ and $\pi_1(y, x) = y$ as the source and target maps d and r, respectively. Taking into account that R is transitive, the composition of elements is defined in the following way: $(y, z)(z, x) = (y, x)$. Let now Γ be a groupoid over X. We say that $x \in X$ remains in the relation with $y \in X$ if and only if there exist $\gamma \in \Gamma$ such that $d(\gamma) = x$ and $r(\gamma) = y$. By using the categorial language we can say that in the groupoid $R \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} X$ there exists a unique morphism (an arrow) from x to y, $x, y \in X$, whereas in the groupoid Γ over X there can exist many morphisms (arrows) from x to y. To obtain the groupoid isomorphic with $R \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} X$ we introduce on Γ the following equivalence relation ρ

Definition 3 Let $\gamma, \xi \in \Gamma$. We say that $\gamma \ \rho \ \xi \Leftrightarrow d(\gamma) = d(\xi), r(\gamma) = r(\xi)$.

Let us make the quotient $\Gamma_\rho := \Gamma/\rho$. The equivalence class of $\gamma \in \Gamma$ is denoted by $[\gamma] \in \Gamma_\rho$, and the natural projection $\pi_\rho : \Gamma \to \Gamma_\rho$ is given by $\pi_\rho(\gamma) = [\gamma]$. On Γ_ρ we introduce the groupoid structure by defining: (a) source and target mappings $d([\gamma]) := d(\gamma)$, $r([\gamma]) := r(\gamma)$; (b) product $[\gamma][\xi] := [\gamma\xi]$; (c) inverse $[\gamma]^{-1} := [\gamma^{-1}]$; (d) inclusion $\varepsilon([d([\gamma])]) := [\varepsilon(d(\gamma))]$.

Proposition 2 *Let Γ be a groupoid over a set X. The groupoid Γ_ρ is isomorphic with the groupoid $R \subset X \times X$ associated with Γ.*

Proof The mapping $f : \Gamma_\rho \to R$ given by $f(a) = (y, x)$ for every $a \in \Gamma_\rho$, such that $d(a) = x, r(a) = y$ is clearly an isomorphism of groupoids. Let us, for instance, consider $a, b \in \Gamma_\rho$ such that $d(a) = x, r(a) = y, r(b) = x, d(b) = z$. Then the product is

$$f(ab) = (y, z) = (y, x)(x, z) = f(a)f(b),$$

and from the construction it follows that f is both surjective and injective. □

3 Reduction of the Left Regular Groupoid Representation

Let us assume that Γ is equipped with a left Haar system of measures [11], and let $X = \Gamma^0$. We consider the Hilbert space $H_x = L^2(\Gamma^x)$, $x \in X$, with the scalar product $< \cdot, \cdot >_x$ (with respect to the corresponding measure), and the bundle of Hilbert spaces $\mathcal{H} = \{H_x\}_{x\in X}$.

We define a unitary operator $U(\gamma) : H_{d(\gamma)} \to H_{r(\gamma)}$ by

$$[U(\gamma)(\psi)](\gamma') = \psi(\gamma^{-1}\gamma'),$$

where $\psi \in L^2(\Gamma^{d(\gamma)}),\ \gamma' \in \Gamma^{r(\gamma)}$. The operator U satisfies the following conditions

(i) $U(\varepsilon(x)) = id|_{H_x}, \forall x \in X$,
(ii) $U(\gamma_1\gamma_2) = U(\gamma_1)U(\gamma_2), \forall(\gamma_1, \gamma_2) \in \Gamma^2$,
(iii) $U(\gamma^{-1}) = U(\gamma)^{-1}$.

The pair $(U, \mathcal{H})$ is said to be the left regular representation of the groupoid Γ over X [11].

Now, we want to transfer the action of the representation $(U, \mathcal{H})$ to a certain subset of functions belonging to $L^2(\Gamma^x)$ so as these functions could be regarded as functions defined on X. This process is called the reduction of the representation $(U, \mathcal{H})$ of the groupoid Γ.

We assume that there exists a function $\alpha : \Gamma \to \mathbb{C}$ with the following properties

(i) $\alpha(\gamma_1\gamma_2) = \alpha(\gamma_1)\alpha(\gamma_2), \forall\gamma_1, \gamma_2 \in \Gamma^{(2)}$,
(ii) $\alpha^*(\gamma) = \alpha(\gamma^{-1})$,
(iii) $\alpha^{-1}(\gamma) = \alpha(\gamma^{-1})$,
(iv) $\alpha(\varepsilon(x)) = 1, \forall x \in X$.

We further assume that the functions $\psi \in L^2(\Gamma^{d(\gamma)})$ satisfy the following condition (called structural condition)

$$\psi(\gamma\ell) = \alpha(\ell^{-1})\psi(\gamma),$$

where $\gamma \in \Gamma$, $\ell \in \Gamma^{d(\gamma)}_{d(\gamma)}$.

Let us define the function $\varphi : \Gamma \to \mathbb{C}$ by

$$\varphi(\gamma) = \alpha(\gamma)\psi(\gamma).$$

For $\ell \in \Gamma^{d(\gamma)}_{d(\gamma)}$ we obtain

$$\begin{aligned}\varphi(\gamma\ell) &= \alpha(\gamma\ell)\psi(\gamma\ell)\\ &= \alpha(\gamma)\alpha(\ell)\alpha(\ell^{-1})\psi(\gamma)\\ &= \alpha(\gamma)\psi(\gamma)\\ &= \varphi(\gamma).\end{aligned}$$

Moreover, from the properties of the function α we have

$$
\begin{aligned}
< \varphi, \varphi >_{d(\gamma)} &= < \alpha\psi, \alpha\psi >_{d(\gamma)} \\
&= < \psi, \psi >_{d(\gamma)} .
\end{aligned}
$$

Hence, $\varphi \in L^2(\Gamma^{d(\gamma)})$.

Let us notice that any two morphisms γ_1, γ_2 such that $\gamma_1 \; \rho \; \gamma_2$ can be written as

$$\gamma_1 = (\gamma_2\gamma_2^{-1})\gamma_1 = \gamma_2(\gamma_2^{-1}\gamma_1),$$

where $\gamma_2^{-1}\gamma_1 \in \Gamma_{d(\gamma)}^{d(\gamma)}$. Consequently,

$$\gamma_1 \; \rho \; \gamma_2 \Rightarrow \varphi(\gamma_1) = \varphi(\gamma_2(\gamma_2^{-1}\gamma_1)) = \varphi(\gamma_2).$$

Any function having this property is said to be consistent with the relation ρ [7]. Therefore, φ determines the function $\widehat{\varphi} : \Gamma_\rho \to \mathbb{C}$ given by

$$\widehat{\varphi}([\gamma]) = \widehat{\varphi} \circ \pi_\rho(\gamma) = \varphi(\gamma),$$

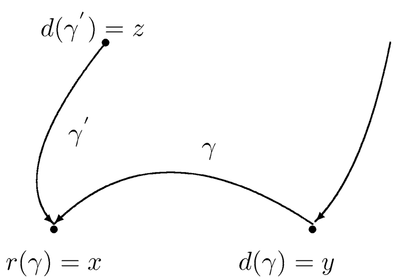
where $\pi_\rho : \Gamma \to \Gamma_\rho$ is the natural projection. We assume that $\widehat{\varphi}_y \in \Gamma_\rho^{d([\gamma])}$ where $d([\gamma]) = y$ is an element of the Hilbert space $L^2(\Gamma_\rho^{d([\gamma])})$.

Now, we transfer the action of the operator U to the functions $\widehat{\varphi}$ according to the following diagram

$$
\begin{array}{ccc}
\psi_y \in L^2(\Gamma^{d(\gamma)}) & \xrightarrow{U} & \psi'_x \in L^2(\Gamma^{r(\gamma)}) \\
\widehat{\ } \circ \alpha \cdot \big\downarrow & & \widehat{\ } \circ \alpha \cdot \big\downarrow \\
\widehat{\varphi}_y \in L^2(\Gamma_\rho^{d([\gamma])}) & \xrightarrow{\widehat{U}} & \widehat{\varphi}'_x \in L^2(\Gamma_\rho^{r([\gamma])}).
\end{array}
$$

Let $\gamma', \gamma \in \Gamma$, and let us consider the situation displayed on Fig. 1.

Fig. 1



With the help of this diagram we compute

$$\begin{aligned}[\bar{U}(\gamma)\widehat{\varphi}]([\gamma']) &= \widehat{\varphi}'([\gamma']) \\ &= \varphi'(\gamma') \\ &= \alpha(\gamma')\psi'(\gamma') \\ &= \alpha(\gamma')\psi(\gamma^{-1}\gamma') \\ &= \alpha(\gamma')\alpha^{-1}(\gamma^{-1}\gamma')\alpha(\gamma^{-1}\gamma')\psi(\gamma^{-1}\gamma') \\ &= \alpha(\gamma')\alpha^{-1}(\gamma^{-1}\gamma')\varphi(\gamma^{-1}\gamma') \\ &= \alpha(\gamma')\alpha^{-1}(\gamma')\alpha^{-1}(\gamma^{-1})\varphi(\gamma^{-1}\gamma') \\ &= \alpha(\gamma)\widehat{\varphi}([\gamma^{-1}\gamma']).\end{aligned}$$

Since the groupoid Γ_ρ is isomorphic with the groupoid associated with the relation R we can write

$$[\gamma'] = (x, z), \quad \gamma^{-1} = (y, x),$$

for γ such that $d(\gamma) = y$, $r(\gamma) = x$ and γ' such that $d(\gamma') = z$, $r(\gamma') = x$. We thus have

$$\begin{aligned}[\gamma^{-1}\gamma'] &= [\gamma^{-1}][\gamma'] \\ &= (y, x)(x, z) \\ &= (y, z).\end{aligned}$$

Hence, we obtain the following formula for the action of the representation $\hat{U}$ on functions $\widehat{\varphi}$

$$[\widehat{U}(\gamma)\widehat{\varphi}](x, z) = \alpha(\gamma)\widehat{\varphi}(y, z).$$

Let us consider $\gamma_1, \gamma_2 \in \Gamma$ such that $d(\gamma_1) = d(\gamma_2) = y$ and $r(\gamma_1) = r(\gamma_2) = x$. The situation is shown on Fig. 2.

We have

$$\begin{aligned}[\widehat{U}(\gamma_1)\widehat{\varphi}](x, z) &= \alpha(\gamma_1)\widehat{\varphi}(y, z), \\ [\widehat{U}(\gamma_2)\widehat{\varphi}](x, z) &= \alpha(\gamma_2)\widehat{\varphi}(y, z).\end{aligned}$$

Hence,

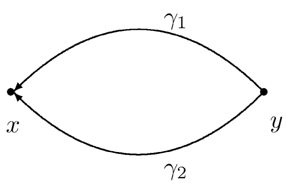
$$\begin{aligned}[\widehat{U}(\gamma_2)\widehat{\varphi}](x, z) &= \alpha(\gamma_2)\alpha^{-1}(\gamma_1)\alpha(\gamma_1)\widehat{\varphi}(y, z) \\ &= \alpha(\gamma_2\gamma_1^{-1})\alpha(\gamma_1)\widehat{\varphi}(y, z) \\ &= \alpha(\gamma_2\gamma_1^{-1})[\widehat{U}(\gamma_1)\widehat{\varphi}](x, z),\end{aligned} \tag{1}$$

where $\gamma_2\gamma_1^{-1} \in \Gamma_x^x$.

The above process is called reduction of the left regular groupoid representation.

From the properties of the function $\alpha : \Gamma \to \mathbb{C}$ it follows that $\alpha|_{\Gamma_x^x}$ is a unitary representation of the isotropy group.

Fig. 2



4 Fundamental Groupoid

In this section we collect some information regarding the concept of fundamental groupoid [11]. Let M be a differential manifold, and $\mathcal{P}(M)$ the set of piecewise smooth paths on M, i.e., the set of piecewise smooth mappings from the interval [0, 1] into M

$$\mathcal{P}(M) = \{\gamma : [0,1] \to M\}.$$

From now on, γ will no longer denote elements of the groupoid Γ but rather the above mappings. We introduce on $\mathcal{P}(M)$ the following equivalence relation: we say that $\gamma \sim \gamma'$ if γ and γ' are homotopic (with fixed endpoints).

Let us consider the set $\Gamma = \mathcal{P}(M)/\sim$, and the natural projection $p : \mathcal{P}(M) \to \Gamma$ given by

$$p(\gamma) = [\gamma].$$

We equip the set Γ with the groupoid structure by defining the source and target mappings in the following way

$$d : \Gamma \to \Gamma_0, \quad d([\gamma]) = \gamma(0),$$
$$r : \Gamma \to \Gamma_0, \quad r([\gamma]) = \gamma(1).$$

The set of composable elements is

$$\Gamma^{(2)} = \{([\gamma], [\gamma']) \in \Gamma \times \Gamma | \gamma(0) = \gamma'(1)),$$

and the composition $[\gamma][\gamma'] = [\gamma\gamma']$ is defined to be

$$\gamma\gamma'(t) = \begin{cases} \gamma(2t-1) & \text{if } 1/2 \le t \le 1, \\ \gamma'(2t) & \text{if } 0 \le t \le 1/2. \end{cases}$$

The groupoid Γ is called fundamental groupoid. The following proposition holds.

Proposition 3 *The fundamental groupoid Γ is a Lie groupoid.*

Proof can be found in [9].

From the fact that Γ is a Lie groupoid, it follows that a system of left Haar measures can be defined on it. The Hilbert space $L^2(\Gamma^x)$, for every $x \in M$, is also well defined.

Let us define the function $\alpha : \Gamma \to \mathbb{C}$. If $\gamma_1, \gamma_2 \in [\gamma]$, we look for a function $\overline{\alpha}$ constant on γ_1 and γ_2, i.e., $\overline{\alpha}(\gamma_1) = \overline{\alpha}(\gamma_2)$. This function determines the function α by

$$\alpha([\gamma]) = \overline{\alpha}(\gamma),$$

where $\gamma \in [\gamma]$. To find the function $\overline{\alpha}$ we consider a closed 1-form $\theta = \sum a_i dx_i$, $d\theta = 0$, where a_i are smooth on M. In such a case

$$\int_{\gamma_1} \theta = \int_{\gamma_2} \theta,$$

where $\gamma_1, \gamma_2 \in [\gamma]$. We finally define the functions $\overline{\alpha}$ and α by

$$\alpha([\gamma]) := \overline{\alpha}(\gamma) := \exp\left(i \int_\gamma \theta \right), \tag{2}$$

where $\gamma \in [\gamma]$.

5 Quasiregular Singularity of a Cosmic String

Let X be a space-time containing a cosmic string [15]. We assume that the cosmic string is situated along the z-axis and its energy-momentum tensor is given by $T_t^t = T_z^z = 2\pi\mu\frac{\delta(r)}{r}$ where $\mu = \frac{1}{4G}(\frac{1-A}{A})$ is a mass density, and $0 < A < 1$. By solving Einstein's equations we obtain the following metric on the space-time X

$$ds^2 = dt^2 - dr^2 - Ar^2 d\theta^2 - dz^2.$$

The isometric transformation of coordinates $\Theta = A\theta$ leads to the flat space-time with the metric [6]:

$$ds^2 = dt^2 - dr^2 - r^2 d\Theta^2 - dz^2,$$

where $0 < \Theta < 2\pi A = 2\pi - \beta < 2\pi$. Locally, X is a Minkowski space, but it contains the singularity at $r = 0$. Such a space-time can be obtained from the usual Minkowski space by removing points $r = 0$ and identifying $(t, r, \phi + k\beta, z) \sim (t, r, \phi, z)$ for $k \in \mathbb{N}$.

Since we are interested in space coordinates of the manifold X, we decompose space-time (locally) into time t and hypersurfaces $t = \text{const}$. We consider one of such hypersurfaces and denote it by M. Let Γ be a fundamental groupoid over M. Since every two points $x, y \in M$ can be connected by a smooth curve, the groupoid Γ is transitive. Consequently, Γ/ρ is isomorphic with the pair groupoid $R = M \times M$ as shown on the following diagram

$$\begin{array}{ccc} \Gamma/\rho & \longleftrightarrow & M \times M \\ d\downarrow\downarrow r & & \pi_1\downarrow\downarrow\pi_2 \\ M & \longleftrightarrow & M \end{array}$$

Wave functions $\widehat{\varphi}$ are thus defined on the entire Cartesian product $M \times M$. We will now construct the function $\alpha : \Gamma \to \mathbb{C}$. Let us consider a 1-form ω on M. In the Cartesian coordinates x^1, x^2, x^3 we have

$$\omega = \frac{-\Phi}{2\pi}\frac{x^2}{r^2}dx^1 + \frac{\Phi}{2\pi}\frac{x^1}{r^2}dx^2,$$

where $\Phi = \text{const}$, $r^2 = (x^1)^2 + (x^2)^2$ and $x^3 = z$. 1-form ω is closed. Let us notice that every closed 1-form on M with quasiregular singularity is cohomologically equivalent to the above 1-form [13]. Let γ be a curve such that $\gamma(0) = z$, $\gamma(1) = x$; from now on $x, z \in M$. From formula (2) we have

$$\alpha(\gamma) = \exp\left(i\int_\gamma \omega_\mu(x^1, x^2, x^3)dx^\mu\right).$$

In the previous section, we have seen that if $\gamma_1\ \rho\ \gamma_2$ then

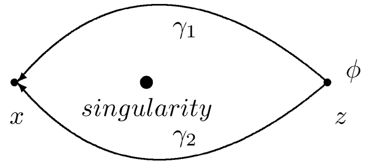
$$\alpha(\gamma_1) = \alpha(\gamma_2) = \alpha([\gamma_1]),$$

where $\gamma_1 \in \Gamma$.

Let us consider a curve γ that encircles the singularity $r = 0$ n-times. Since the integral $\exp(i\int_\gamma \omega_\mu(x^1, x^2, x^3)dx^\mu)$ does not depend on the shape of the curve γ, we perform our computations for the circle C of a radius R centered at $r = 0$

$$\int_\gamma \omega_\mu dx^\mu = \int_C \omega_\mu dx^\mu = \Phi n\left(1 - \frac{\beta}{2\pi}\right).$$

Fig. 3



Hence, we obtain

$$\alpha(\gamma) = \exp\left\{ i\Phi n\left(1 - \frac{\beta}{2\pi}\right)\right\}. \tag{3}$$

Wave functions $\widehat{\varphi}$ on M depend on two variables, $\widehat{\varphi}(x,z)$, $x, z \in M$. By fixing $z \in M$ for all pairs $(x,z) \in M \times M$ we obtain a wave function depending on one variable; we denote it by $\phi(x) = \widehat{\varphi}(x,z)$.

Let us consider the action of the operator $\widehat{U}$ on functions ϕ. The situation is presented on Fig. 3.

For this situation, from formula (1) we have

$$\begin{aligned}[\widehat{U}(\gamma_2)\phi](x) &= \alpha(\gamma_2\gamma_1^{-1})[\widehat{U}(\gamma_1)\phi](x) \\ &= \exp\left\{ i\Phi\left(1 - \frac{\beta}{2\pi}\right)\right\}[\widehat{U}(\gamma_1)\phi](x). \end{aligned} \tag{4}$$

Here $n = 1$ since $\gamma_2\gamma_1^{-1}$ is a loop that encircles the singularity only once.

From formula (4) it follows that the phase of the wave function on M depends on the total curvature of the manifold M contained in the cosmic string (in the angle defect β). It is a gravitational counterpart of the Aharonov-Bohm effect.

We can also look at this effect in the following way. The groupoid Γ over M naturally acts on the manifold M. Let

$$\Gamma \star M := \{(\gamma, x) \subset \Gamma \times M \mid d(\gamma) = x\}.$$

The left action of Γ on M is a mapping $\Gamma \star M \to M$ denoted by $(\gamma, x) \mapsto \gamma x$, and given by $\gamma x = r(\gamma)$. It has the following properties

(ii) $(\gamma\xi)x = \gamma(\xi x)$,
(iii) $(\varepsilon(x))x = x$.

We now consider the position observable $\hat{x}$ in the presence of the cosmic string. Let us first write the wave function $\widehat{\varphi}(x,z)$ as

$$\widehat{\varphi}_x := \widehat{\varphi}(x,z).$$

It belongs to $H_x = L^2(R^x)$. The support of $\widehat{\varphi}_x$ consists of elements of the pair groupoid $R = M \times M$ that end at x. As we can see, $\hat{U}$ satisfies the conditions of the groupoid representation, i.e., it acts between elements of different Hilbert spaces indexed by $x \in M$. This is shown in the following diagram

$$\begin{array}{ccc} y \in M & \xrightarrow{\gamma} & x \in M \\ \downarrow & & \downarrow \\ \widehat{\varphi}_y \in H_y & \xrightarrow{\widehat{U}} & (\widehat{U}(\gamma)\widehat{\varphi})_x \in H_x \end{array}$$

Let us notice that for different γ_1, γ_2, such that $x = \gamma_1 y = \gamma_2 y$, we obtain the wave functions $(\widehat{U}(\gamma_1)\widehat{\varphi})_x$, $(\widehat{U}(\gamma_2)\widehat{\varphi})_x$ with different phases (given by (3)).

If the considered system is symmetric with respect to the actions of the groupoid Γ, then

$$\Phi\left(1 - \frac{\beta}{2\pi}\right) = 2\pi m$$

for $m \in \mathbb{Z}$. From this it follows that the constant Φ can assume only discrete values

$$\Phi = \frac{2\pi m}{(1 - \frac{\beta}{2\pi})}. \tag{5}$$

This means that the representation $\alpha|_{\Gamma_x^x}$ is trivial for every $x \in M$.

This result can be interpreted in the following way. If Φ is different from the one given by (5) then the interference occurs (there is a phase difference), and the position measurement is impossible. If, however, Φ is given by (5), there is no phase difference and the position measurement is possible.

To sum up, if a physical system, in the presence of a cosmic string, is symmetric with respect to the action of the groupoid Γ, we do not observe the Aharonov-Bohm effect, and we are able to measure the position of a particle (there is no interference of wave functions). However, if such a symmetry does not exist, the phase difference causes the interference of wave functions, and the latter depends on the total curvature of the manifold contained in the quasiregular singularity of the cosmic string. The effect is purely gravitational since, in spite of the fact that the particle moves in a flat region of space-time around the cosmic string, it behaves differently than in the Minkowski space-time.

References

1. Bezerra, V.B.: Ann. Phys. (New York) **203**, 392 (1990)
2. Bezerra, V.B.: Class. Quantum Gravit. **8**, 1939 (1991)
3. Corichi, A., Pierri, M.: Phys. Rev. D **51**, 5870 (1995)
4. Ford, L.H., Vilenkin, A.: J. Phys. A **14**, 2353 (1981)
5. Gheorghe, I.J.: Novi Sad J. Math. **31**, 23 (2002)
6. Heller, M.: Int. J. Theor. Phys. **31**, 5840 (1992)
7. Heller, M., Sasin, W., Trafny, A., Żekanowski, Z.: Acta Cosmol., Krakow **18**, 57 (1992)
8. Ho, V.B., Morgan, M.J.: Aust. J. Phys. **47**, 245 (1994)
9. Mach-Stadler, M.: J. Math. Sci. **113**, 637 (2003)
10. Mikami, K., Weinstein, A.: Publ. RIMS Kyoto Univ. **42**, 121–140 (1988)
11. Paterson, A.L.T.: Groupoids, Inverse Semigroups and Their Operator Algebras. Birkhäuser, Berlin (1998)
12. dos Santos, I., Bezerra, V.B.: Braz. J. Phys. **23**, 100 (1993)
13. Schwartz, A.S.: Quantum Field Theory and Topology. Springer, Berlin (1993)
14. Sitenko, Y., Mishchenko, A.: J. Exp. Theor. Phys. **81**, 831 (1995)
15. Vickers, J.A.G.: Class. Quantum Gravit. **4**, 1 (1987)